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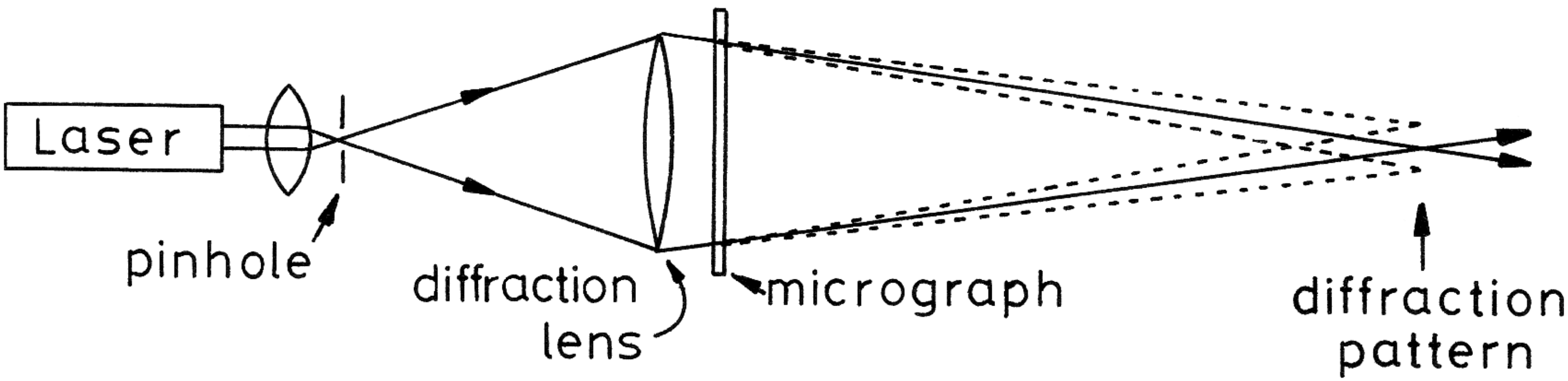
BASIC PRINCIPLES OF FOURIER THEORY (M.F. Moody)

- Some history.
- What is a Fourier transform (F.T.)?
- Working with F.Ts.: 1D curves.
- Fourier analysis of 2D images.

Early History of Fourier Transforms & Image Processing.

- 1810: Fourier invented F.Ts. for heat conduction problems.
- 1873: Abbe applied them to image formation in the microscope.
- 1939: Bragg applied them to X-ray crystallography.
- 1949: Lipson produced them with the optical diffractometer.
- 1964: Klug & Berger used optical diffractometer on electron micrographs.
- 1965: Cooley & Tukey re-invented Fast Fourier Transform (Gauss, 1805) for computers.
- 1968: DeRosier & Klug used computed F.Ts. for **3-D reconstruction.**
- 1971: Erickson & Klug used them for **defocus correction.**

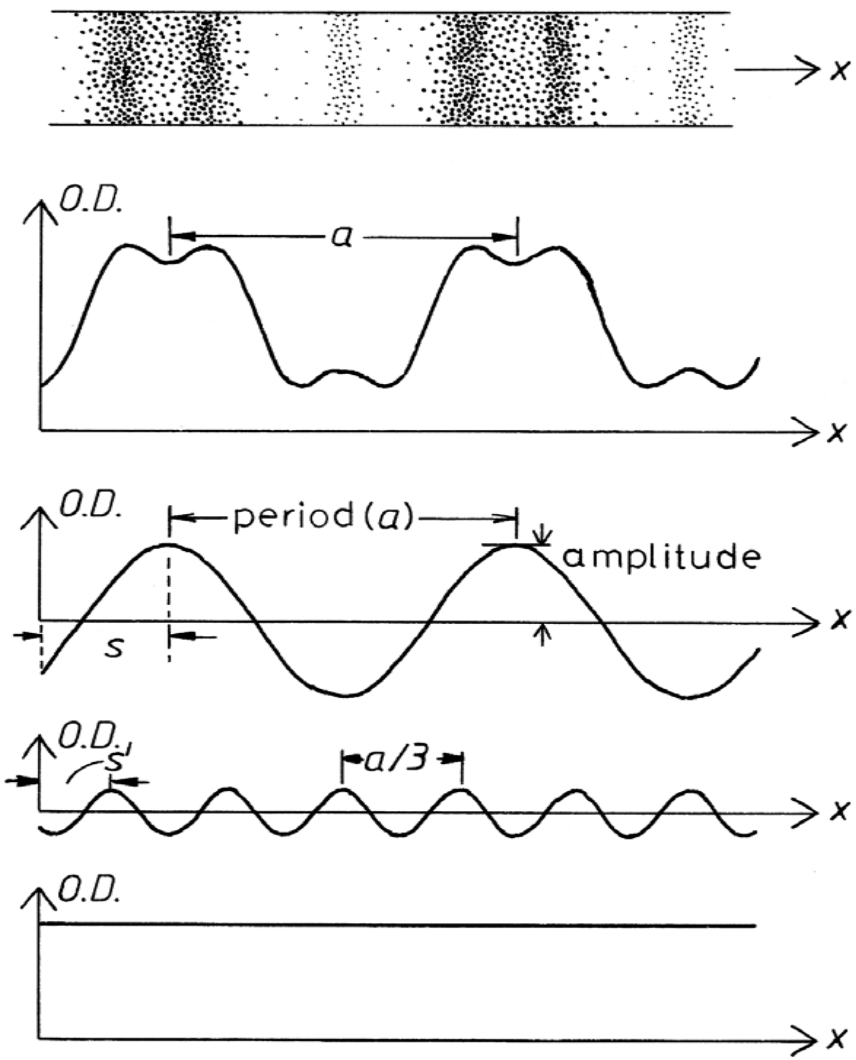
Optical Diffractometer



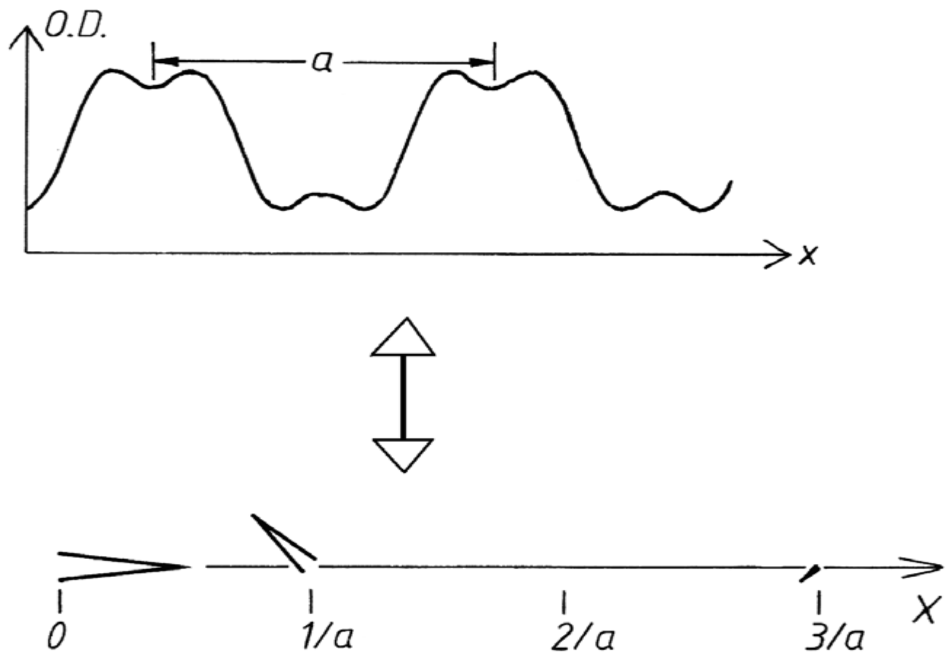
WHAT IS A FOURIER TRANSFORM (F.T.)?

- Fourier series of repeating (periodic) curves.
- Curve represents light passing through transparency: it has brightness and phase.
- Fourier transforms of non-repeating curves (also with brightness and phase).

Periodic Curves & Fourier Series.

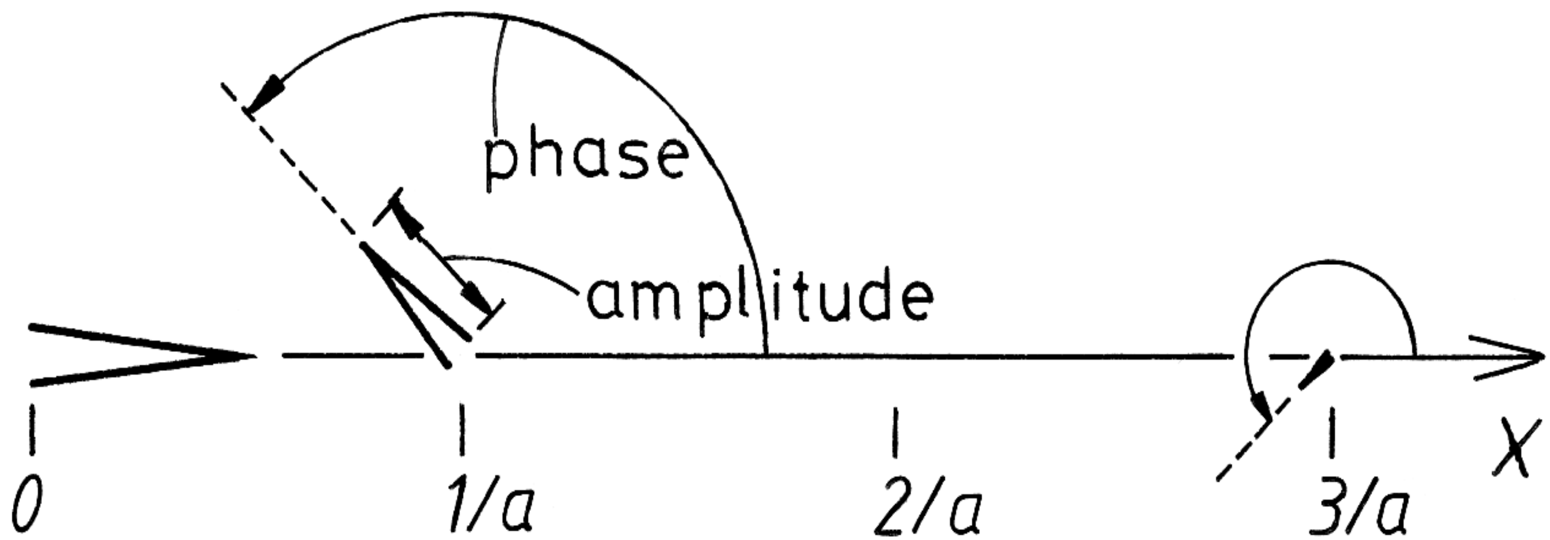


Fourier Series is Reversible.



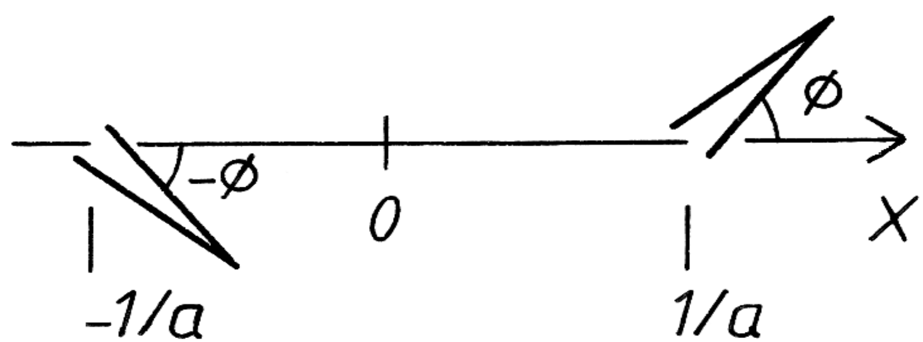
- 3 peaks can reconstitute essentials of curve.
- Remaining “peaks” are just noise.
- So ignoring them = **data-compression**.

Representing Fourier Series.

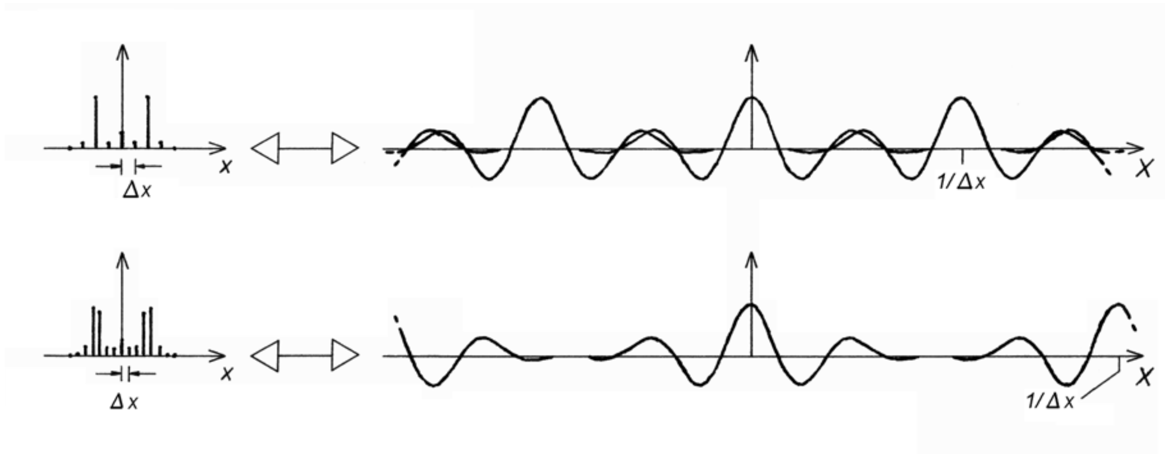


Representation when Curve has Phase as well as Amplitude.

- Light wave has amplitude and **phase**.
- Optical diffractometer gives its F.T.
- This has negative as well as positive axis.



Fourier analysis of non-periodic curves.



- Repeating curve gives Fourier series with 4 peaks.
- Take part of curve and repeat it with gaps between.
- This gives Fourier series with more (& closer) peaks.
- Increase size of gaps until only the central curve exists.
- Then the “peaks” of the Fourier “series” are continuous.
- This gives us the Fourier transform.

Fourier series and F.Ts.

- **Fourier series**: real-space curve is **continuous**, reciprocal-space Fourier series is **discontinuous** (peaks).
- **Fourier transform**: real-space curves is **continuous**, reciprocal-space F.T. is also **continuous**.
- Fourier transforms can be obtained in the optical diffractometer.
- But (strictly) neither the Fourier series nor the F.T. is obtainable with a computer.

WORKING WITH FOURIER TRANSFORMS (F.Ts.).

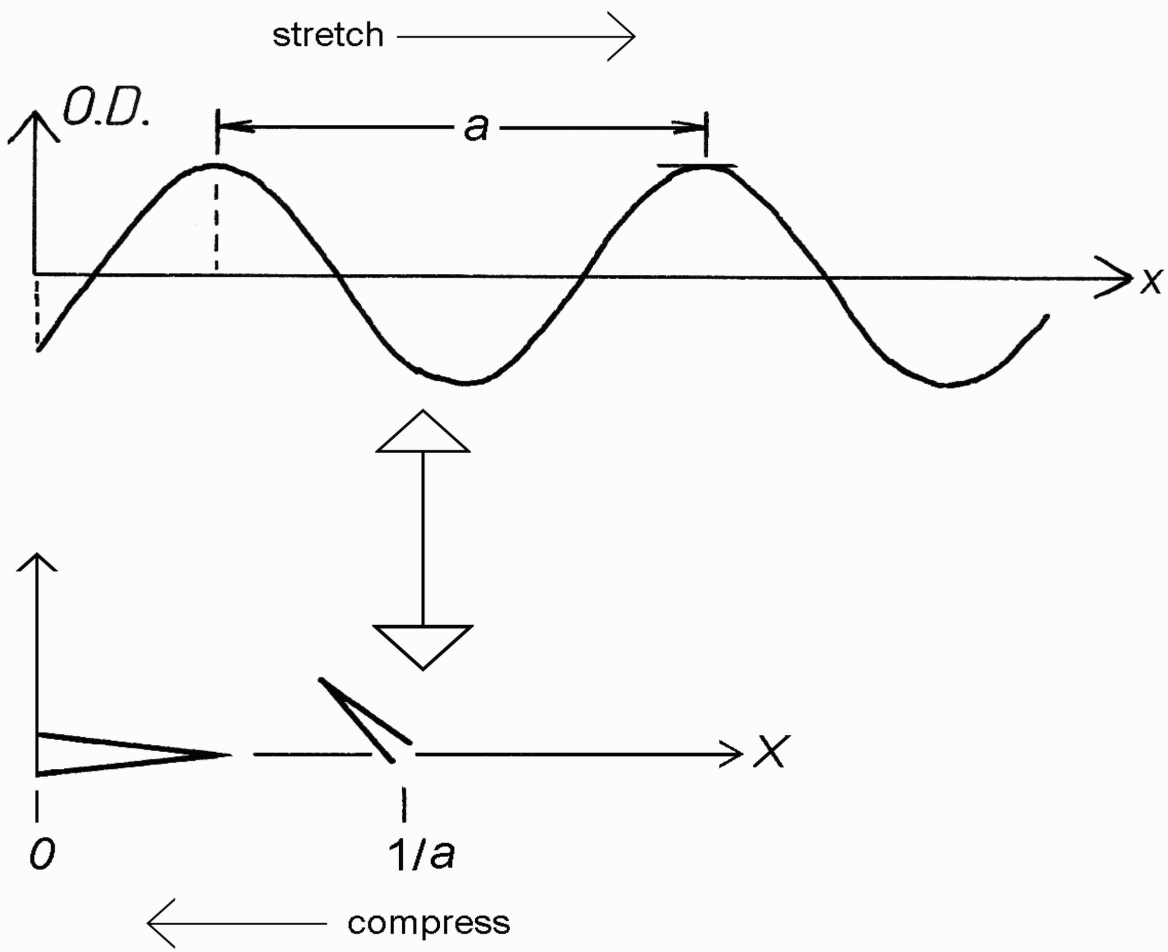
- Reversing F.Ts.
- Rules for 1D F.Ts.
- Examples of 1D F.Ts.
- Rules for 2D F.Ts.: the 3 pairs of rules.
- Examples of 2D F.Ts.

Reversing F.Ts.

- F.T. extends to infinity, so it needs truncation before reversing it.
- Outer parts relate to finer details, so eliminate details that are artefacts or noise.

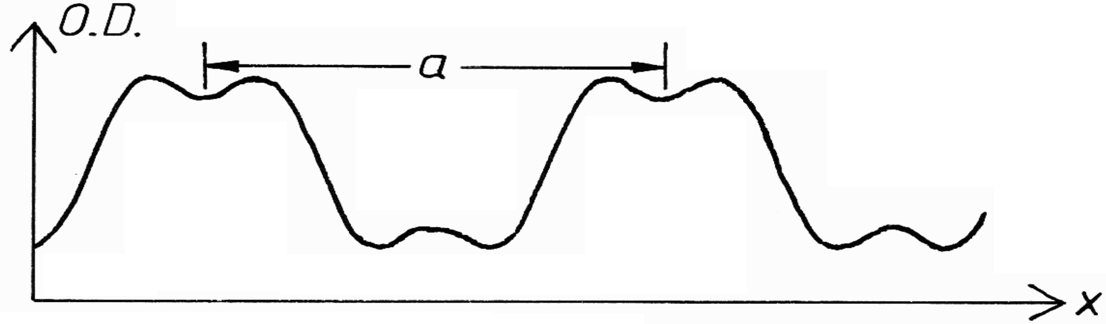
- How to reverse F.T.?
- Simply **take a second F.T.**, and then **rotate it by 180 degrees.**

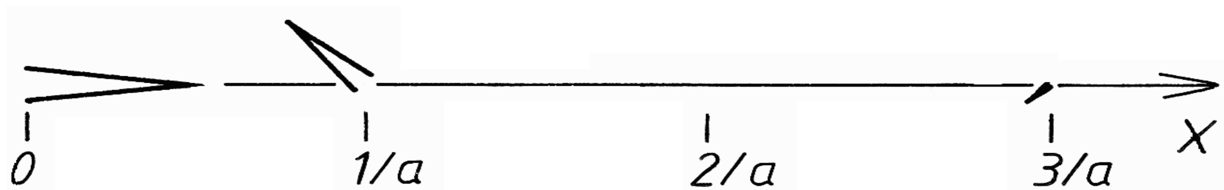
1st. Rule for 1D F.Ts.: Stretching.



2nd. Rule for 1D F.Ts.: Addition.

Sum of Fourier components = (cleaned) O.D. curve:



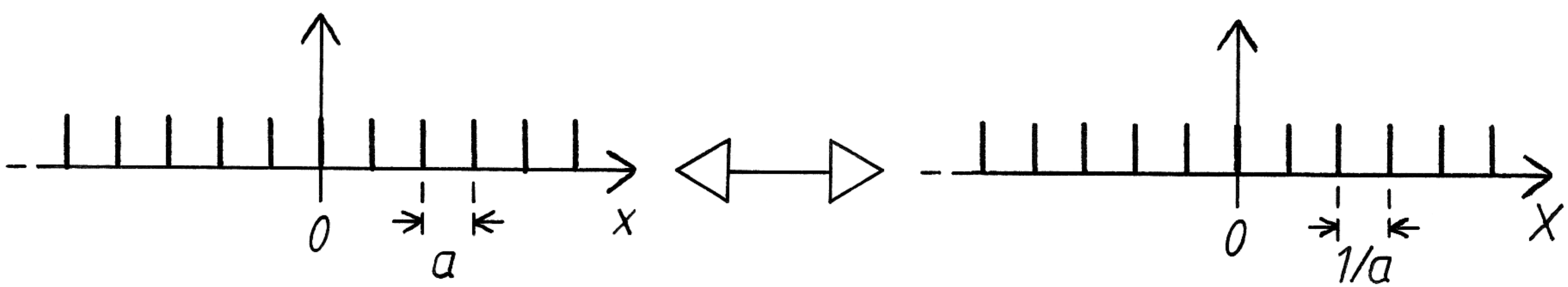


= sum of F.T. peaks corresponding to each Fourier component.

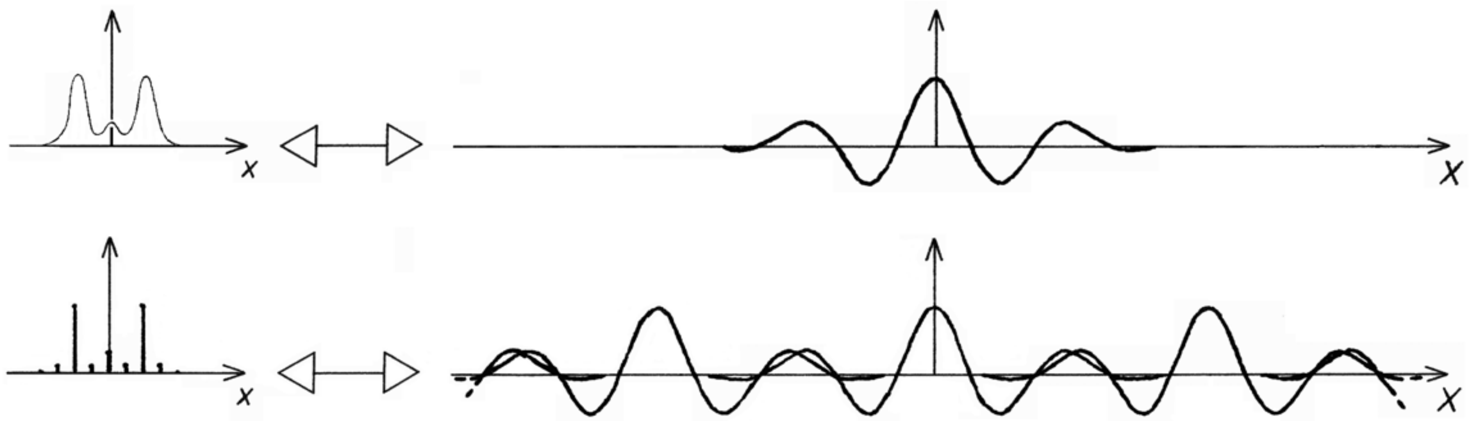
3rd. Rule for 1D F.Ts.: Multiplication.

- Multiplication by a constant → multiplication of F.T. by a constant (*follows from addition rule*).
- Multiplication of 2 images → **convolution** of 2 F.Ts.
- This means that the first F.T. acts as the "laser beam" generating the second F.T.
- Example when first F.T. = set of peaks:-

1D F.T. of infinite equidistant peaks.



1D Convolution.

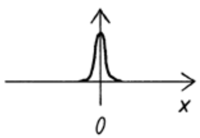


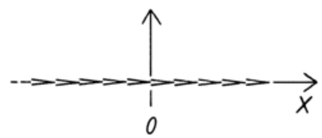
- Curve multiplied by infinite set of peaks, giving 9 peaks of “sampled” curve.
- F.T. gets convoluted by F.T.(peaks), *i.e.* by the reciprocal set of peaks.

4^{th}. Rule for 1D F.Ts.: translation (shift) only changes F.T. phases.

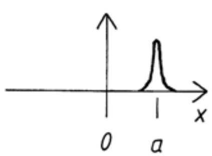
- Moving an image doesn’t change it, so it still needs the same component density-waves to construct it.
- Therefore translation (shift) doesn’t affect F.T. **amplitudes**.
- Therefore the translation only changes the **phases** of its component density-waves.
- A high-frequency density-wave must move as far as one of low-frequency; but its wavelength is shorter, so its proportional shift (i.e. its phase-shift) must be bigger.

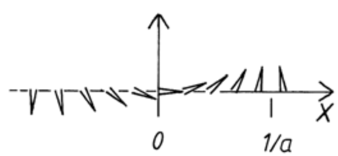
Translating a single peak.





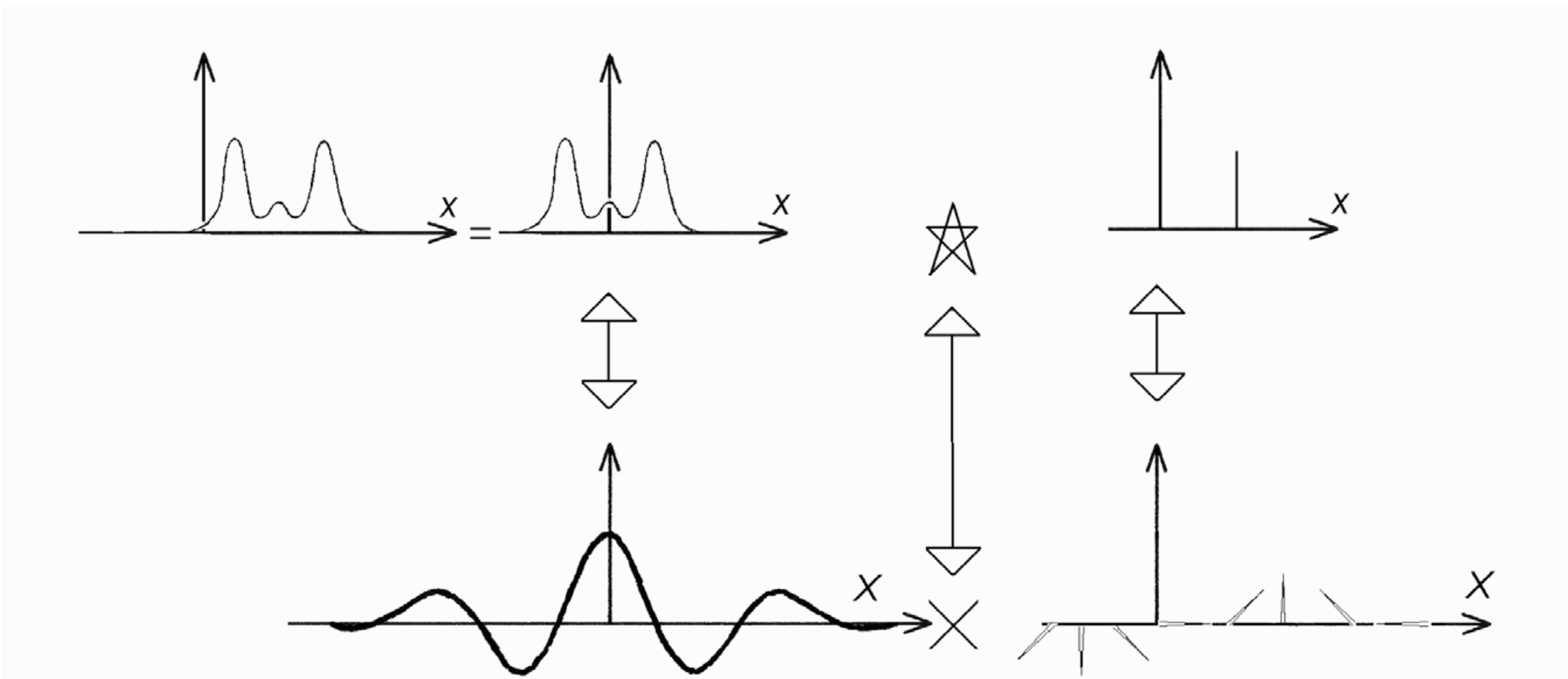
- Peak at origin needs cosine waves of all frequencies, but same amplitude.



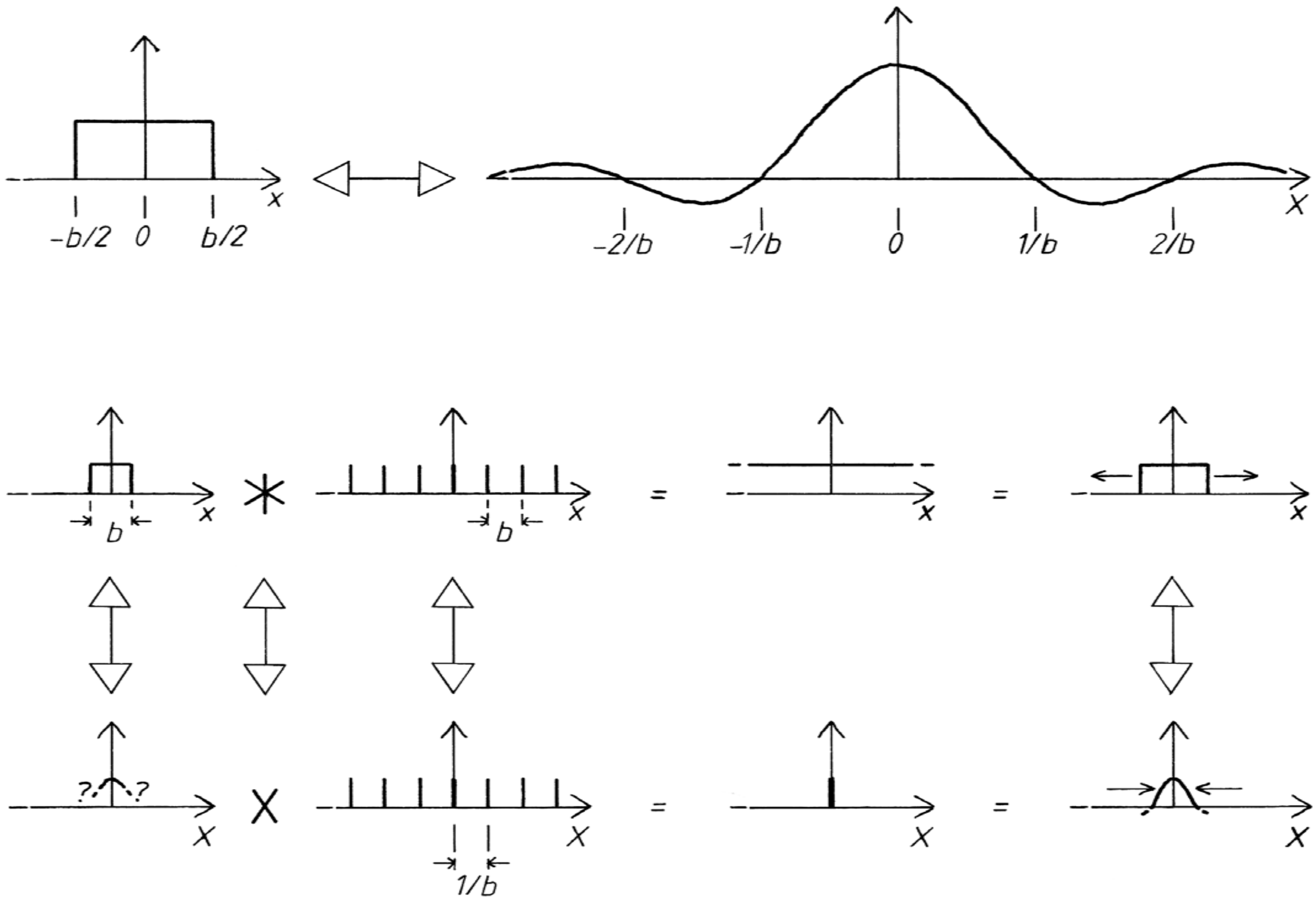


- Peak shifted from origin needs cosine waves with phase proportional to frequency.

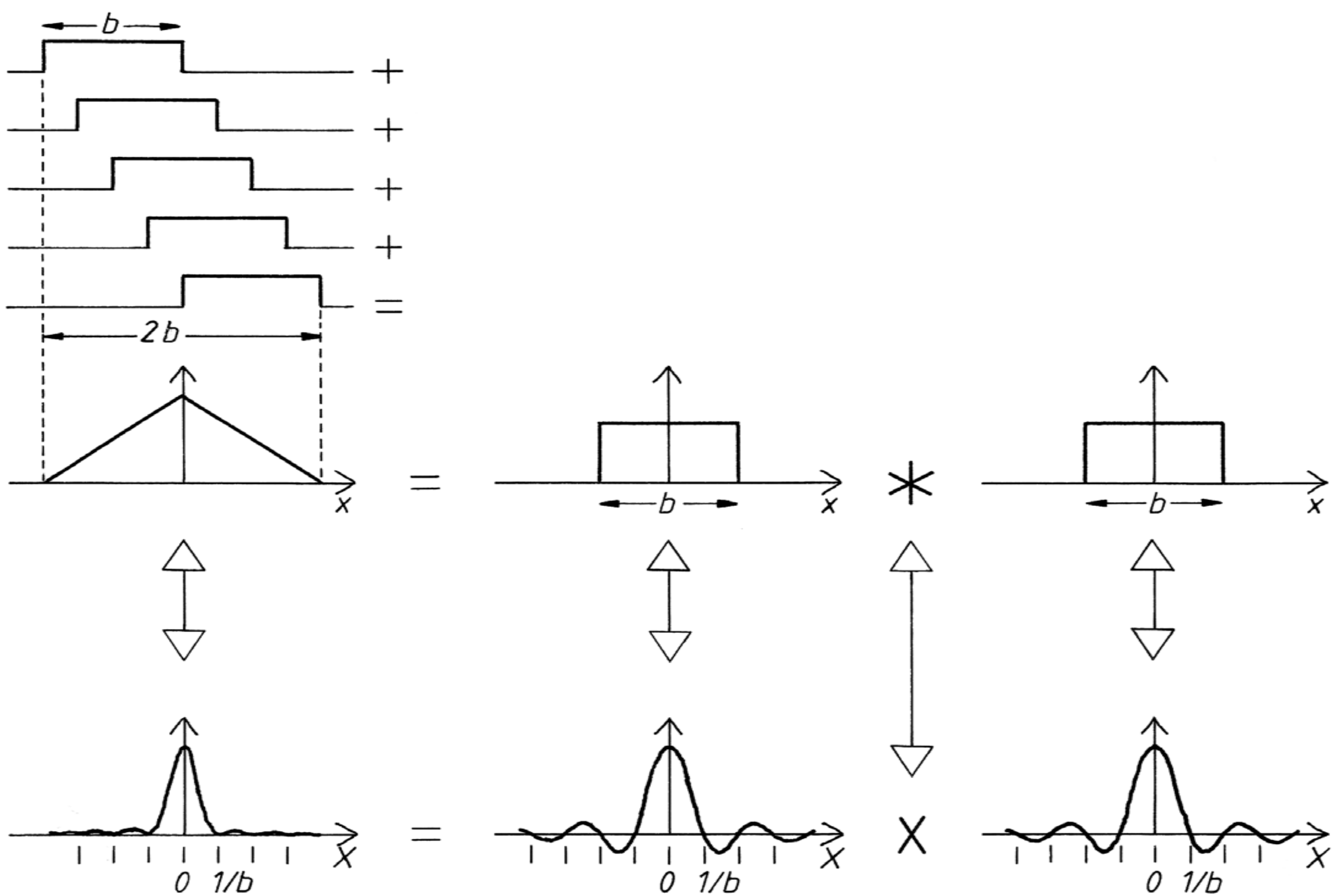
Translating an image.



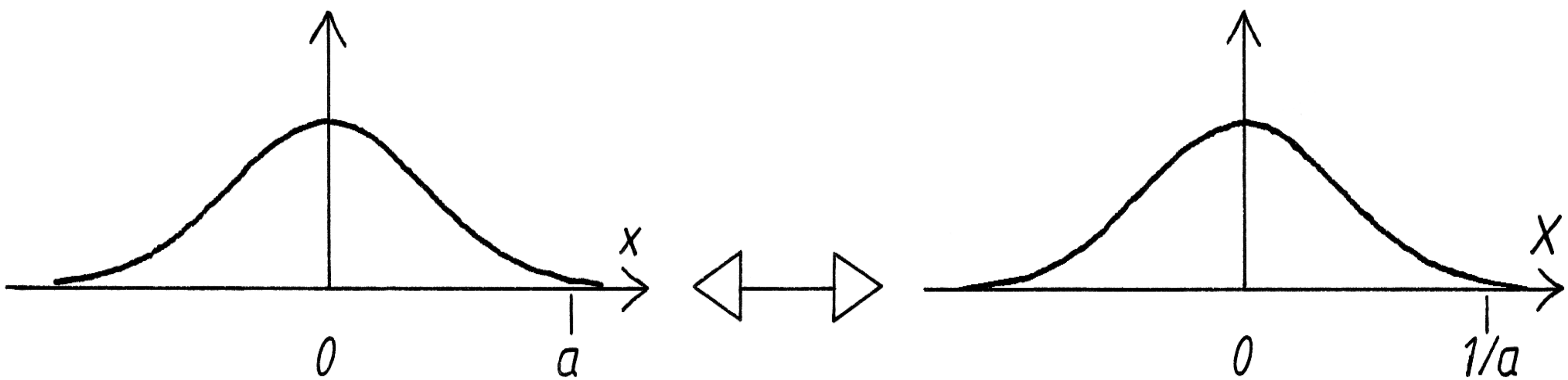
1D F.T.: “rectangle” function.



1D F.T.: “triangle” function.



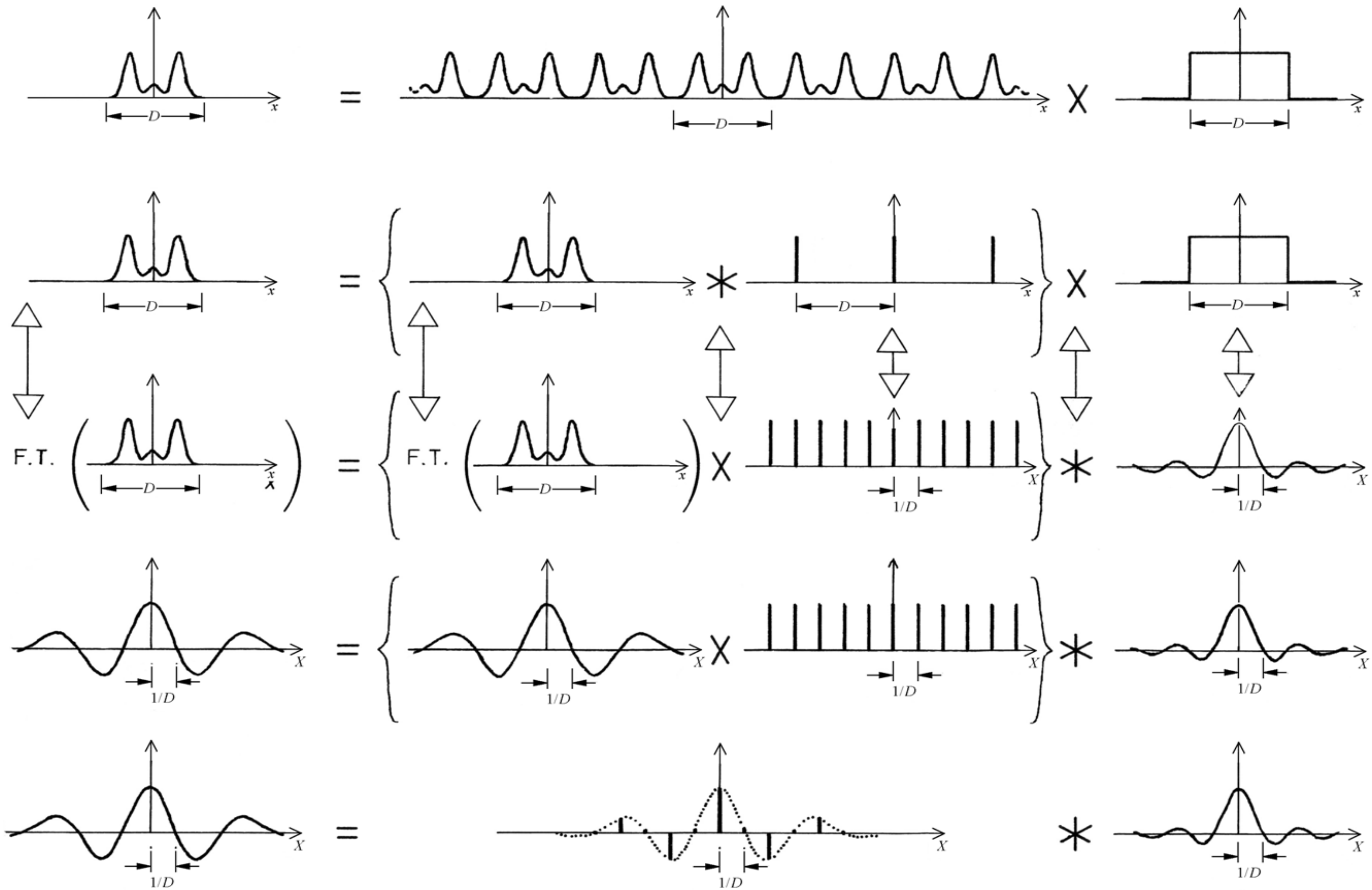
1D F.T.: Gaussian function.



How much information in a F.T.?

- F.Ts. are smooth functions, so they are somewhat predictable.
- By the scale theorem, the smaller an image, the more stretched its F.T., so the more predictable it is.
- Being predictable, an F.T. carries a limited amount of information per unit of reciprocal space.
- How limited is this information?

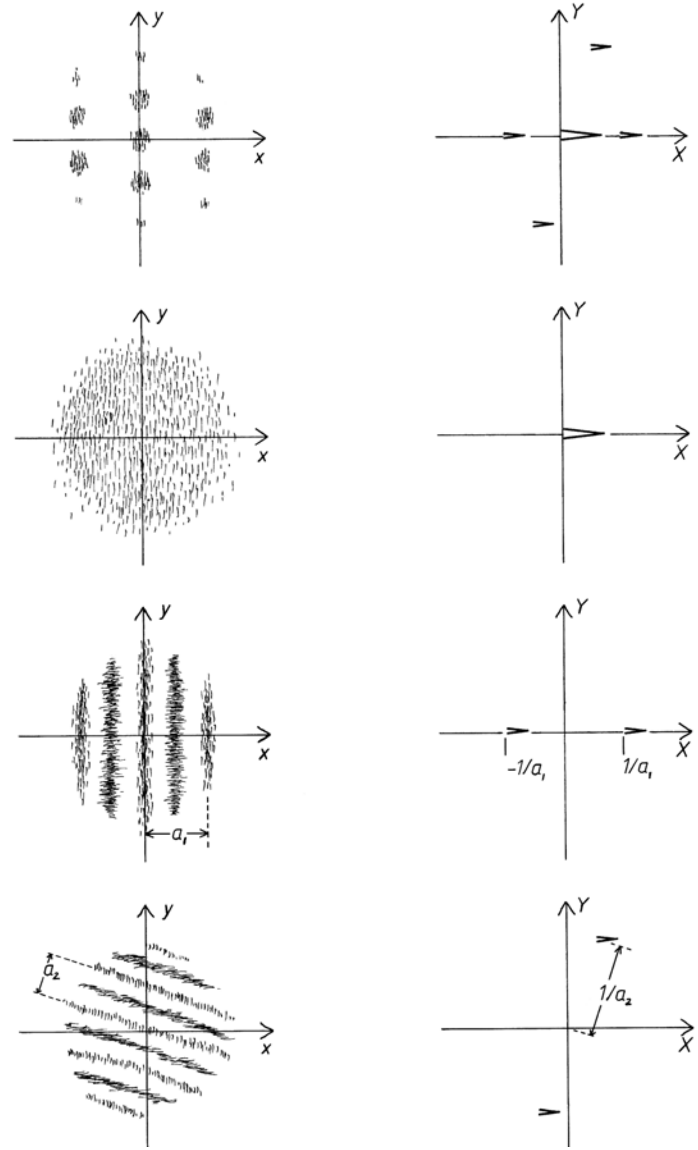
Sampling theorem.



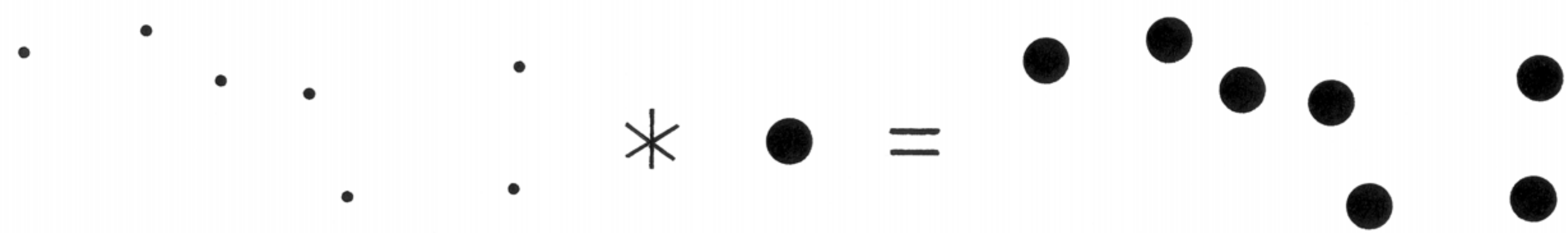
Lessons of sampling theorem.

- The F.T. of a curve of width *D* can be rebuilt accurately from its values sampled at points 1/*D* apart.
- Thus these few sampled values carry all the information in the F.T.
- The F.T. of a curve of width *D* consists of “lumps” of substantial amplitude that are roughly 1/*D* in size.

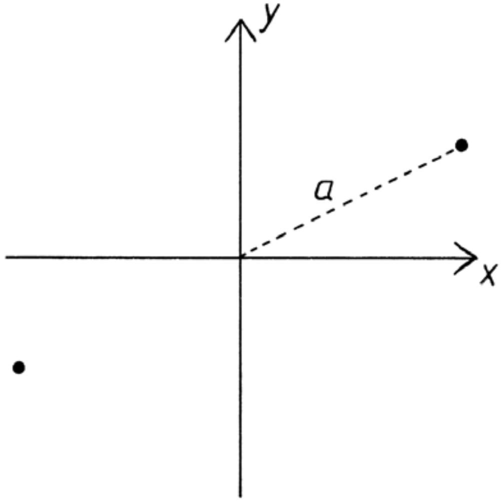
Fourier analysis of 2D images.

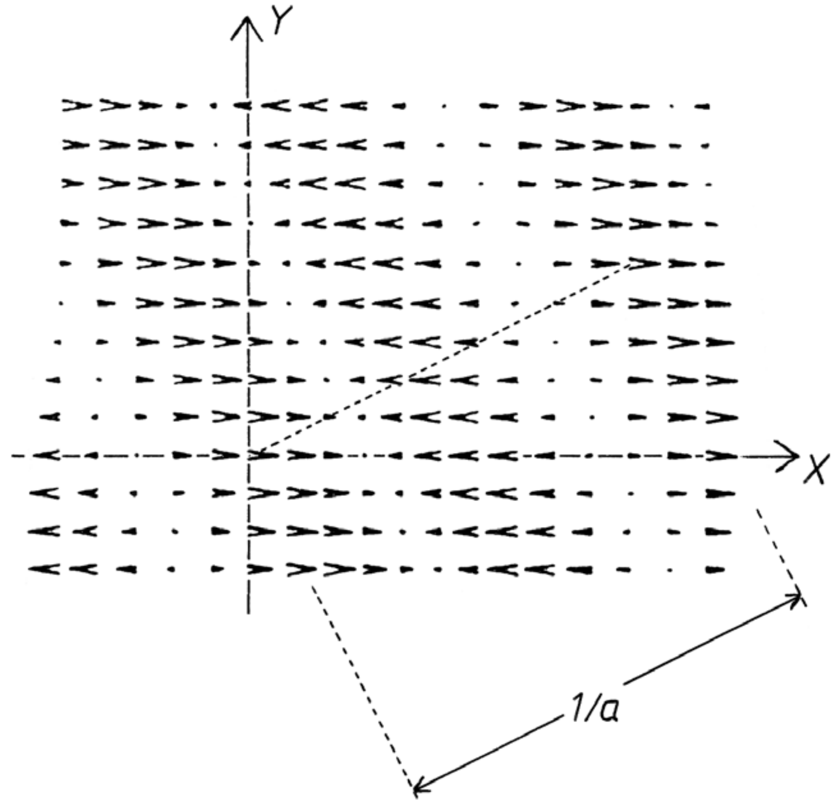


2D convolution.

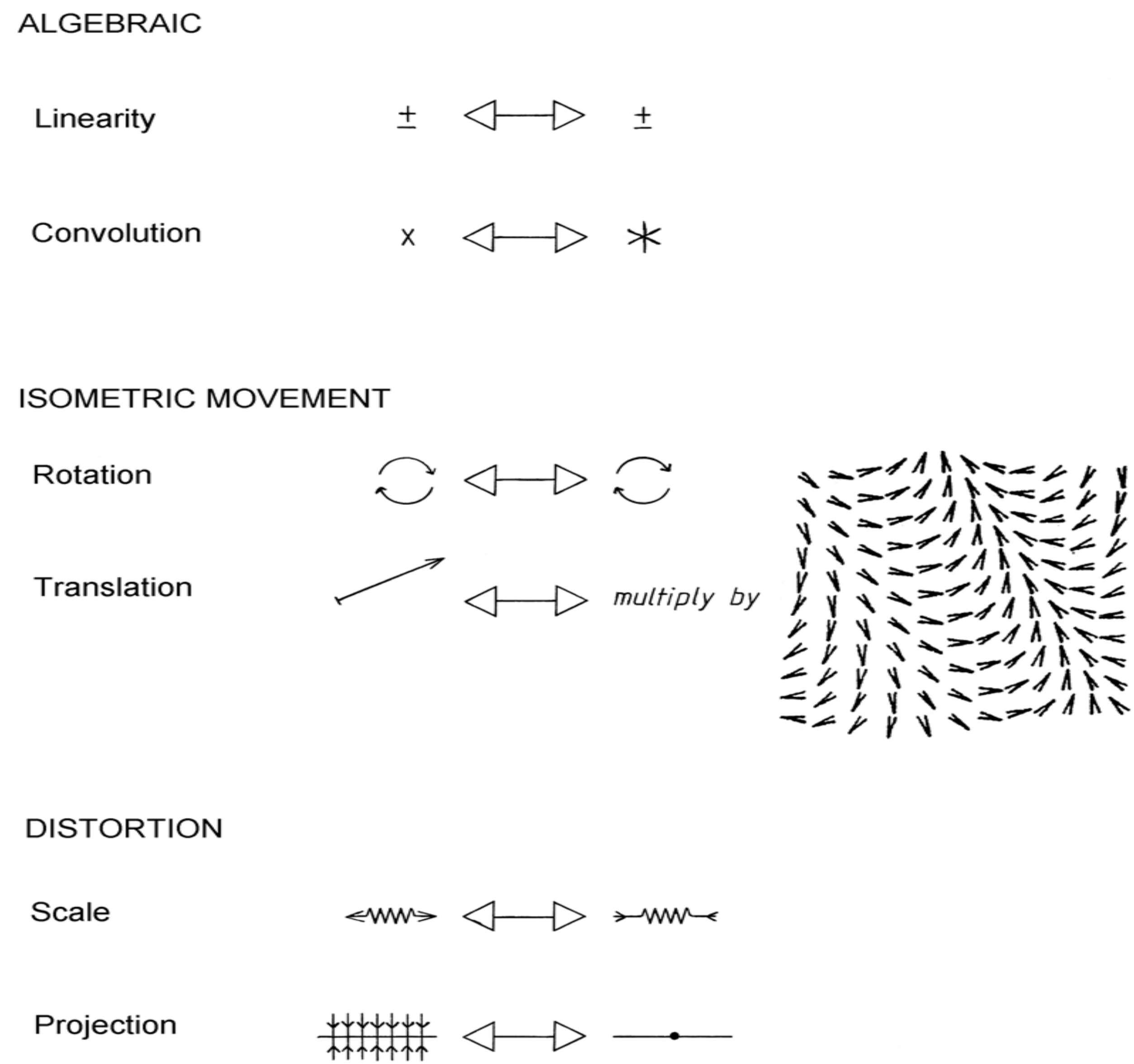


2D F.Ts.: translation.

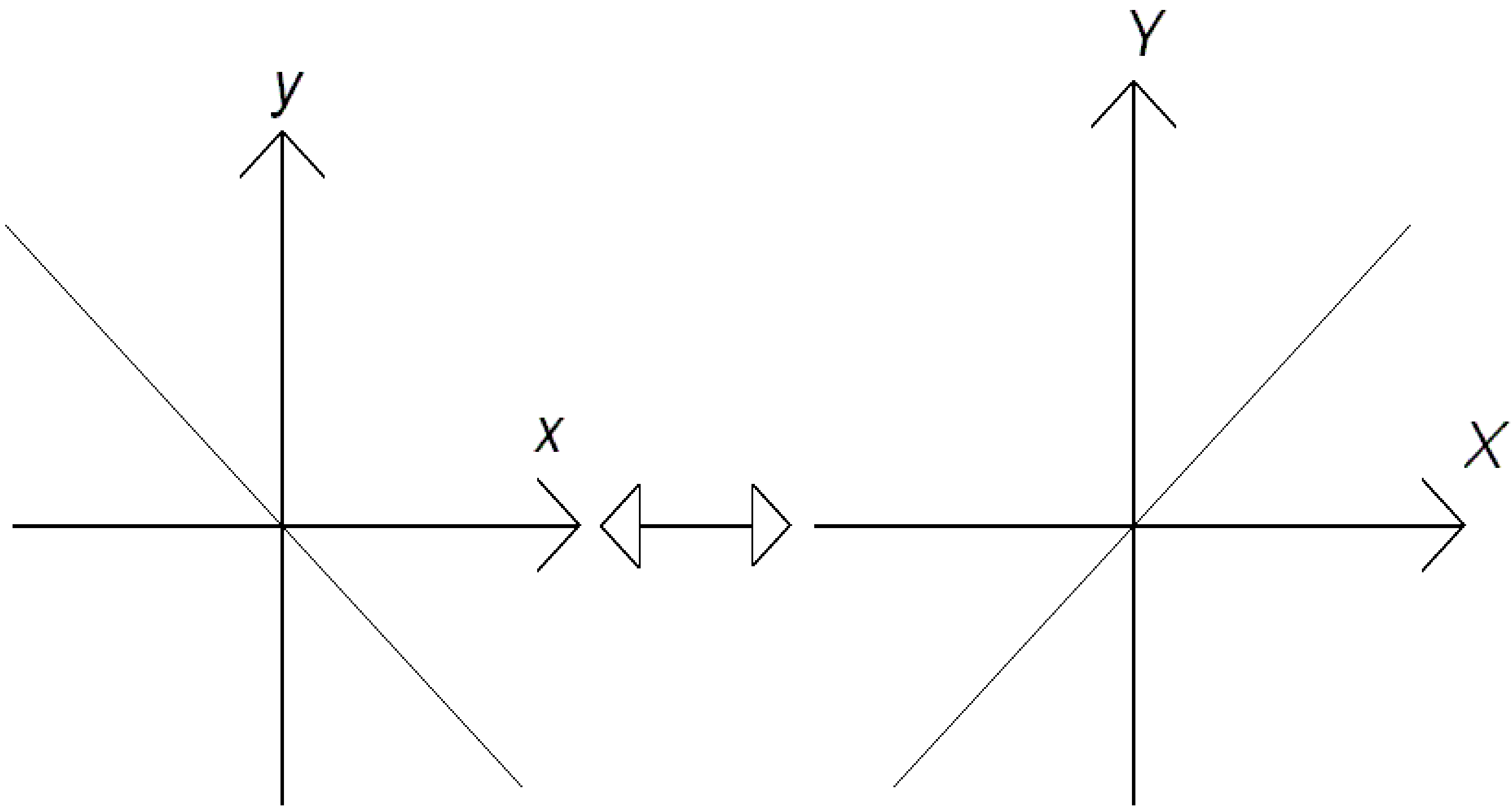




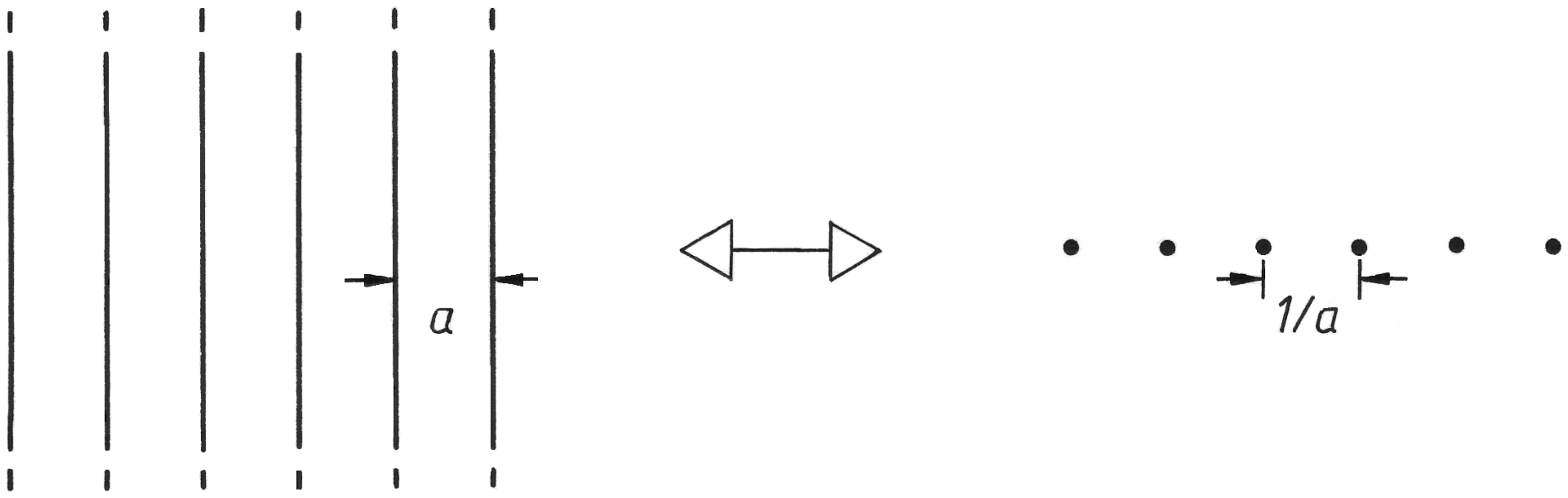
2D F.Ts.: 3 pairs of rules.



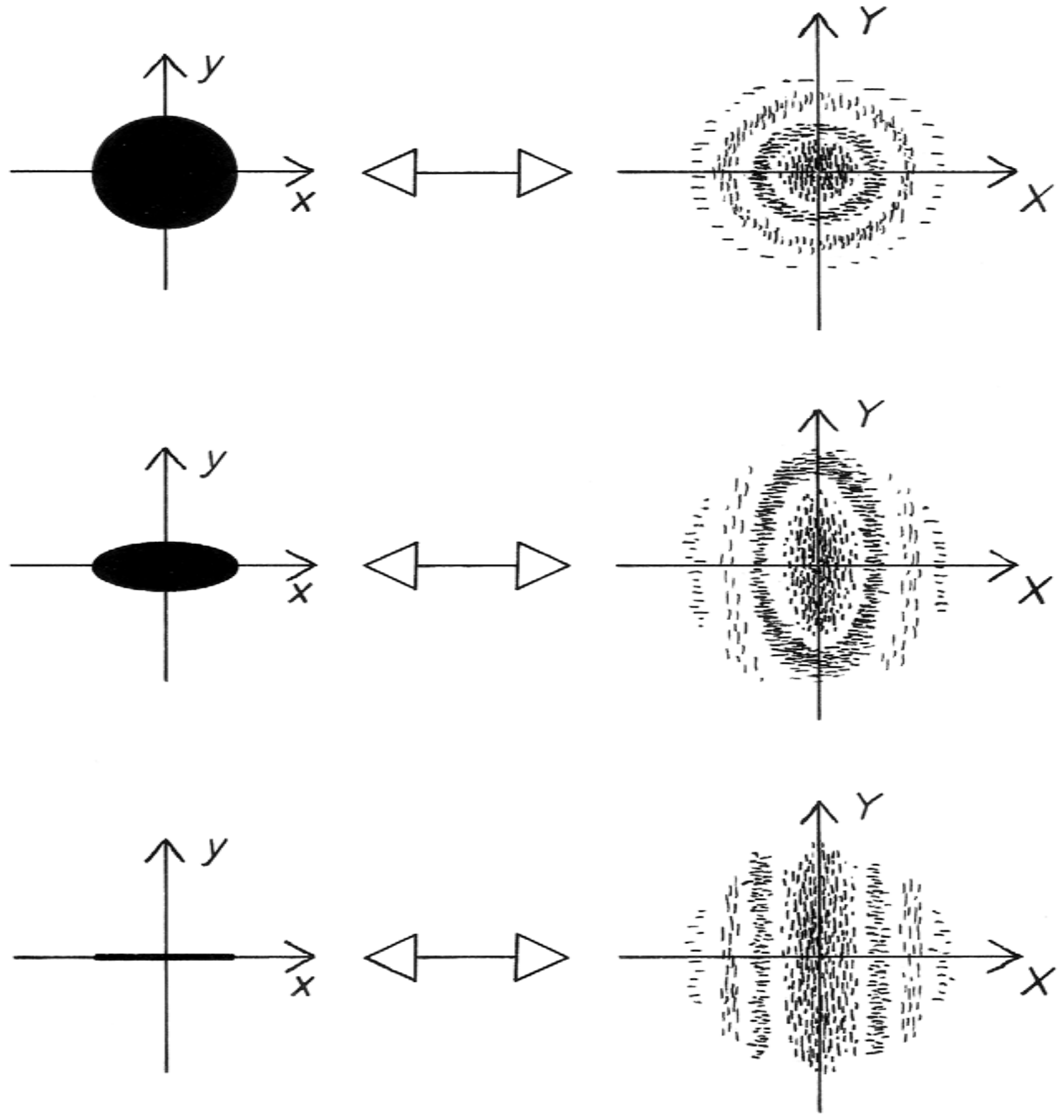
2D F.Ts.: a line.



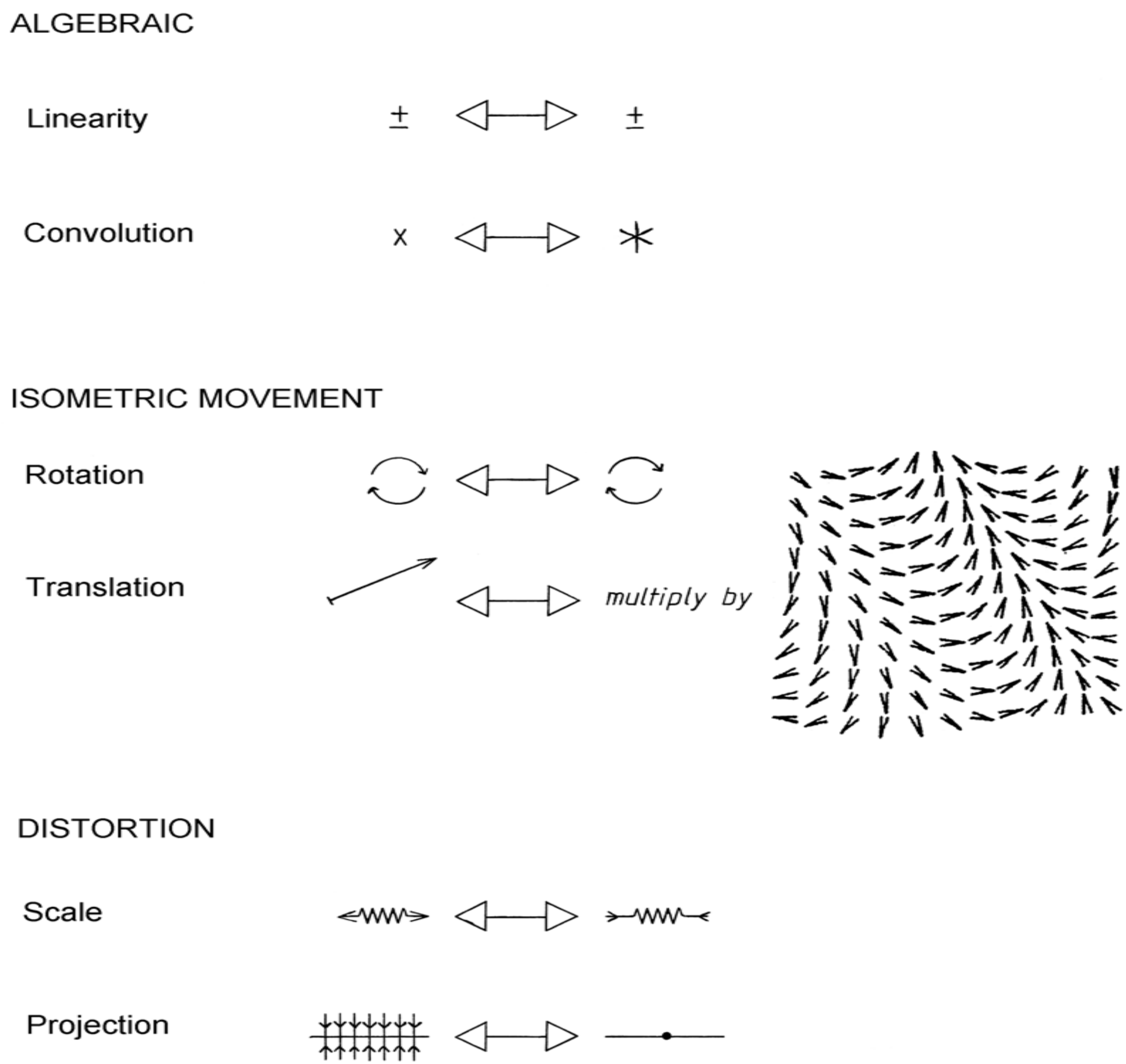
2D F.Ts.: parallel lines.



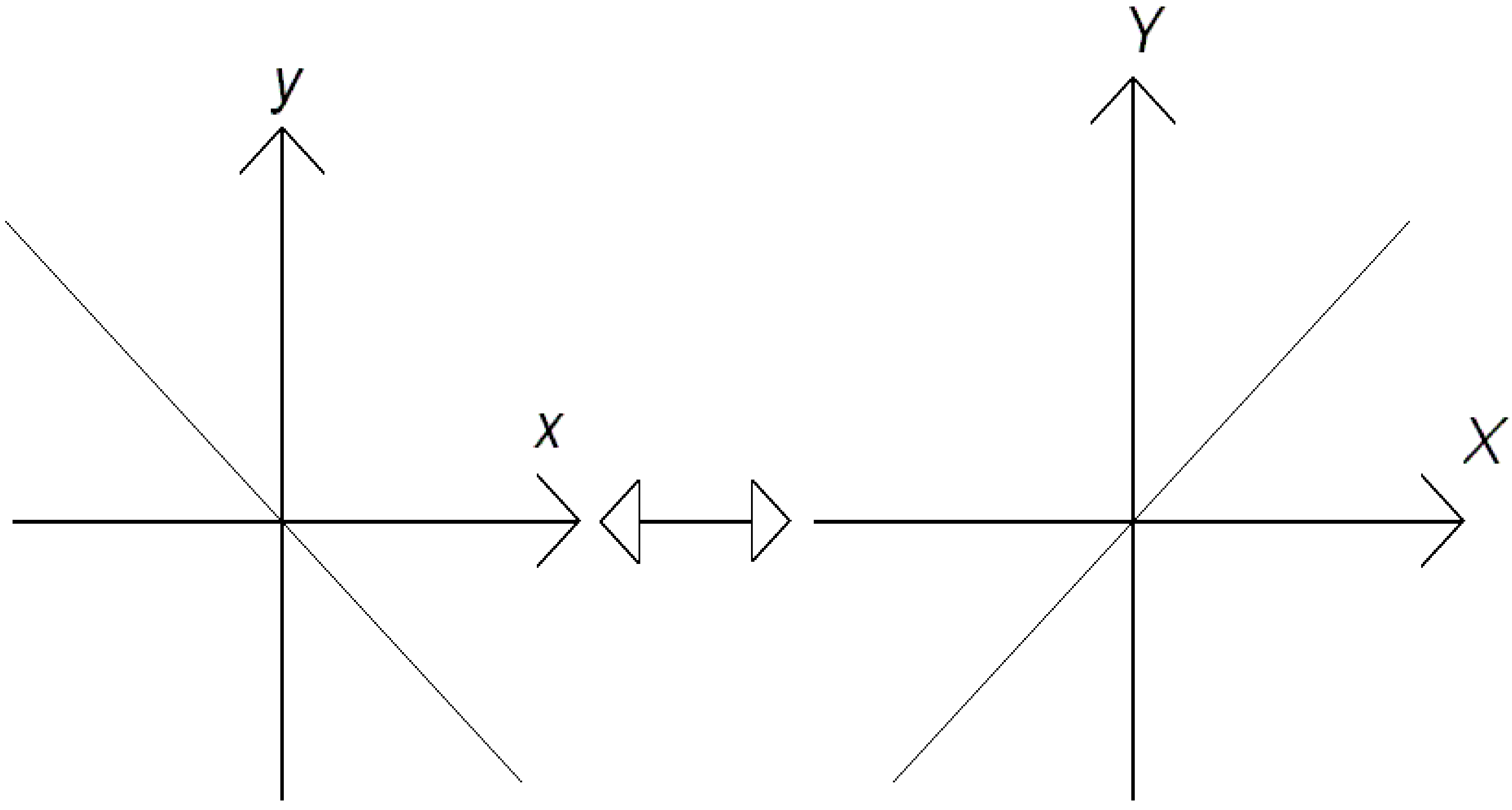
2D F.Ts.: Projection Theorem.



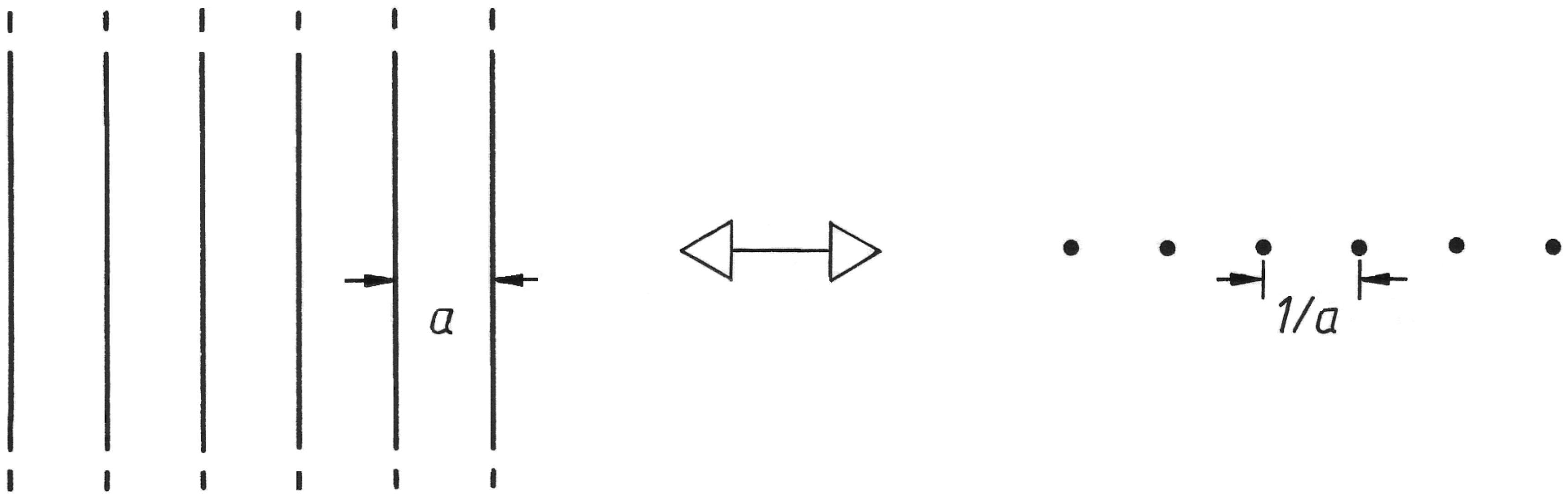
2D F.Ts.: 3 pairs of rules.



2D F.Ts.: a line.



2D F.Ts.: parallel lines.



2D F.Ts.: square lattice.